ON THE RESIDUE CLASS DISTRIBUTION OF THE NUMBER OF PRIME DIVISORS OF AN INTEGER

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Abstract. The *Liouville function* is defined by $\lambda(n) := (-1)^{\Omega(n)}$ where $\Omega(n)$ is the number of prime divisors of n counting multiplicity. Let $\zeta_m := e^{2\pi i/m}$ be a primitive m–th root of unity. As a generalization of Liouville's function, we study the functions $\lambda_{m,k}(n) := \zeta_m^{k\Omega(n)}$. Using properties of these functions, we give a weak equidistribution result for $\Omega(n)$ among residue classes. More formally, we show that for any positive integer m, there exists an $A > 0$ such that for all $j = 0, 1, \ldots, m-1$, we have

$$\#\{n \le x : \Omega(n) \equiv j \pmod{m}\} = \frac{x}{m} + O\left(\frac{x}{\log^A x}\right).$$

Best possible error terms are also discussed. In particular, we show that for $m > 2$ the error term is not $o(x^\alpha)$ for any $\alpha < 1$.

1. Introduction

The *Liouville function*, denoted $\lambda(n)$, is defined by $\lambda(n) := (-1)^{\Omega(n)}$ where $\Omega(n)$ is the number of prime divisors of n counting multiplicity. The Liouville function is intimately connected to the Riemann zeta function and hence to many results and conjectures in prime number theory. Recall that [5, pp. 617–621] for $\Re s > 1$, we have

$$\sum_{n\ge 1} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}, \tag{1}$$

so that $\zeta(s) \ne 0$ for $\Re s \ge \vartheta$ provided

$$\sum_{n \le x} \lambda(n) = o(x^\vartheta).$$

The prime number theorem allows the value $\vartheta = 1$, so that for $j = 0, 1$ we have that

$$\#\{n \le x : \Omega(n) \equiv j \pmod{2}\} \sim \frac{x}{2}.$$

We generalize this result to the following theorem.

Theorem 1.1. *Let m be a positive integer and $j = 0, 1, \ldots, m-1$. Then the (natural) density of the set of all $n \in \mathbb{Z}_{>0}$ such that $\Omega(n) \equiv j \pmod{m}$ exists, and is equal to $1/m$; furthermore, there exists an $A > 0$ such that*

$$N_{m,j}(x) := \#\{n \le x : \Omega(n) \equiv j \pmod{m}\} = \frac{x}{m} + O\left(\frac{x}{\log^A x}\right).$$

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In order to prove this theorem, we study a generalization of Liouville's function. Namely, let m be a positive integer and $\zeta_m := e^{2\pi i/m}$ be a primitive m–th root of unity. Define

$$\lambda_{m,k}(n) := \zeta_m^{k\Omega(n)}.$$

As with $\lambda(n)$, since $\Omega(n)$ is completely additive, $\lambda_{m,k}(n)$ is completely multiplicative. For $\Re s > 1$, denote

$$L_{m,k}(s) := \sum_{n\geq 1} \frac{\lambda_{m,k}(n)}{n^s}.$$

The functions $\lambda_{m,k}(n)$ and $L_{m,k}(s)$ were introduced by Kubota and Yoshida [4]. They gave (basically) a multy-valued analytic continuation of $L_{m,k}(s)$ to the region $\Re s > 1/2$. Using this, for $m \geq 3$ and $k = 1, \ldots, m-1$ with $m/k \neq 2$, they showed that certain asymptotic bounds on the partial sums

$$S_{m,k}(x) := \sum_{n\leq x} \lambda_{m,k}(n),$$

cannot hold; in particular, this sum cannot be $o(x^\alpha)$ for any $\alpha < 1$. Finally, this is used by the authors to show, given Theorem 1.1, that if $m \geq 3$, then an asymptotic of the form

$$N_{m,j}(x) = \frac{x}{m} + o(x^\alpha) \tag{2}$$

cannot hold simultaneously for all $j = 0, 1, \ldots, m-1$, if $\alpha < 1$. We will show that if $m \geq 3$, then for *all* $j = 0, 1, \ldots, m-1$ the asymptotic (2) does not hold if $\alpha < 1$. This is in striking contrast to the expected result for $m = 2$. Recall that in the case that $m = 2$, if the Riemann hypothesis is true then

$$N_{2,j}(x) = \frac{x}{2} + o(x^{1/2+\varepsilon})$$

for $j = 0, 1$ and any $\varepsilon > 0$.

2. Perliminary results

Lemma 2.1. *Let m be a positive integer. Then for $k = 0, 1, \ldots, m-1$, we have*

$$S_{m,k}(x) = \sum_{j=0}^{m-1} \zeta_m^{jk} N_{m,j}(x) \tag{3}$$

and for $j = 0, 1, \ldots, m-1$, we have

$$N_{m,j}(x) = \frac{1}{m} \sum_{k=0}^{m-1} \zeta_m^{-jk} S_{m,k}(x). \tag{4}$$

Proof. We have

$$\begin{aligned} S_{m,k}(x) &= \sum_{n\leq x} \zeta_m^{k\Omega(n)} \\ &= \sum_{j=0}^{m-1} \sum_{\substack{n\leq x \\ \Omega(n)\equiv j \pmod m}} \zeta_m^{k\Omega(n)} \\ &= \sum_{j=0}^{m-1} \zeta_m^{kj} N_{m,j}(x), \end{aligned}$$

which proves the first formula of the lemma. Instead of directly inverting the matrix determined by this formula, we proceed as follows to obtain the second formula. Consider the right–hand side of (4). Using the definition of $\lambda_{m,k}(n)$ we have

$$\frac{1}{m}\sum_{k=0}^{m-1}\zeta_m^{-jk}\sum_{n\le x}\lambda_{m,k}(n)=\sum_{n\le x}\frac{1}{m}\sum_{k=0}^{m-1}\zeta_m^{(\Omega(n)-j)k}.$$

If n satisfies $\Omega(n)\equiv j \pmod m$, then $\zeta_m^{\Omega(n)-j}=1$, so that

$$\frac{1}{m}\sum_{k=0}^{m-1}\zeta_m^{(\Omega(n)-j)k}=1.$$

If n does not satisfy $\Omega(n)\equiv j \pmod m$, then $\zeta_m^{\Omega(n)-j}\neq 1$. We thus have

$$\frac{1}{m}\sum_{k=0}^{m-1}\zeta_m^{(\Omega(n)-j)k}=\frac{1}{m}\cdot\frac{\zeta_m^{(\Omega(n)-j)m}-1}{\zeta_m^{(\Omega(n)-j)}-1}=\frac{1}{m}\cdot\frac{0}{\zeta_m^{(\Omega(n)-j)}-1}=0.$$

This proves the second part of the lemma. □

To yield our density result on the number of prime factors, counting multiplicity, modulo m, we need the following result.

Theorem 2.2. *For every $m\in\mathbb{Z}_{>0}$ there is an $A>0$ such that for all $k=1,\dots,m-1$, we have*

$$|S_{m,k}(x)|\ll\frac{x}{\log^A x}.$$

To prove this, we use the following theorem.

Theorem 2.3 (Hall [3])**.** *Let D be a convex subset of the closed unit disk in $\mathbb{C}$ containing 0 with perimeter $L(D)$. If $f:\mathbb{Z}_{>0}\to\mathbb{C}$ is a multiplicative function with $|f(n)|\le 1$ for all $n\in\mathbb{Z}_{>0}$ and $f(p)\in D$ for all primes p, then*

$$\frac{1}{x}\left|\sum_{n\le x}f(n)\right|\ll\exp\left(-\frac{1}{2}\left(1-\frac{L(D)}{2\pi}\right)\sum_{p\le x}\frac{1-\Re f(p)}{p}\right). \tag{5}$$

Proof. This is a direct consequence of Theorem 1 of [3]. □

Proof of Theorem 2.2. Set D equal to the convex hull of the m–th roots of unity and $f=\lambda_{m,k}$. Because D is a convex subset strictly contained in the closed unit disk of $\mathbb{C}$, we have $L(D)<2\pi$. This gives

$$c:=\frac{1}{2}\left(1-\frac{L(D)}{2\pi}\right)>0.$$

Applying Theorem 2.3 yields

$$\begin{aligned}\frac{1}{x}\left|\sum_{n\le x}\lambda_{m,k}(n)\right|&\ll\exp\left(-c\sum_{p\le x}\frac{1-\Re\lambda_{m,k}(p)}{p}\right)\\&=\exp\left(-c(1-\Re\zeta_m^k)\sum_{p\le x}\frac{1}{p}\right)\end{aligned}$$

Since $\sum_{p\le x} p^{-1} = \log\log x + O(1)$, this quantity is

$$\begin{aligned}&\ll \exp\left(-c(1-\Re\zeta_m^k)\log\log x\right)\\ &= \left(\frac{1}{\log x}\right)^{c(1-\Re\zeta_m^k)}.\end{aligned}$$

Noting that $0<k<m$, we have $c(1-\Re\zeta_m^k)>0$. Set $A := \min_{0<k<m}\{c(1-\Re\zeta_m^k)\}$. Then $A>0$ and we obtain

$$\left|\sum_{n\le x}\lambda_{m,k}(n)\right| \ll \frac{x}{\log^A x}.$$ □

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Lemma 2.1 directly gives us

$$N_{m,j}(x) = \frac{1}{m}S_{m,0}(x) + \frac{1}{m}\sum_{k=1}^{m-1}\zeta_m^{-jk}S_{m,k}(x). \tag{6}$$

The first term of the right-hand side (6) is

$$\frac{1}{m}S_{m,0}(x) = \frac{1}{m}\sum_{n\le x}1 = \frac{x}{m}+o(1).$$

Applying the triangle inequality and Theorem 2.2, we get that the absolute value of the second term of the right-hand side of (6) is

$$\left|\frac{1}{m}\sum_{k=1}^{m-1}\zeta_m^{-jk}S_{m,k}(x)\right| \le \frac{1}{m}\sum_{k=1}^{m-1}|S_{m,k}(x)| \ll \frac{x}{\log^A x}$$

for some $A>0$. This gives us our desired result. □

4. Results for error terms

For $m\in\mathbb{Z}_{>0}$ and $j=0,1,\dots,m-1$, we introduce the error term

$$R_{m.j}(x) := N_{m,j}(x) - \frac{x}{m}.$$

Theorem 1.1 implies that

$$R_{m,j}(x) = o(x).$$

For $m>2$, Kubota and Yoshida [4] prove, conditionally on Theorem 1.1, that at least one of the error terms $R_{m,j}(x)$ is not $o(x^\alpha)$ for any $\alpha<1$. We strengthen their result (unconditionally) as follows.

Theorem 4.1. *Let $m\in\mathbb{Z}_{>2}$ and let $\alpha<1$. Then none of $R_{m,0},R_{m,1},\dots,R_{m,m-1}$ are $o(x^\alpha)$.*

Following [4], we use the following results.

Lemma 4.2. *Let $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ be a sequence of complex numbers and $\alpha>0$. If the partial sums satisfy $\sum_{n\le x}a_n = o(x^\alpha)$, then the Dirichlet series $\sum_{n\ge1}a_n n^{-s}$ converges for $\Re s>\alpha$ to a holomorphic (single–valued) function.*

Proof. This follows directly from Perron's formula [1, p. 243 Lemma 4]. □

Theorem 4.3. *Let $m \in \mathbb{Z}_{>2}$ and let $k = 1, 2, \ldots, m-1$. The Dirichlet series $L_{m,k}(s)$ can be analytically continued to a multi–valued function on $\Re s > 1/2$ given by the product $\zeta(s)^{\zeta_m^k} G_{m,k}(s)$ where $G_{m,k}(s)$ is an analytic function for $\Re s > 1/2$. In particular, if $k \neq m/2$, then for any $\alpha < 1$ the Dirichlet series $L_{m,k}(s)$ does not converge for all s with $\Re s > \alpha$.*

Proof. The first part follows from Theorem 1 in [4] (strictly speaking this handles only the case $k = 1$, but the proof of this theorem works for general k). Note that ζ_m^k is not rational for $k \neq m/2$. Since $\zeta(s)$ has a pole at $s = 1$, this mens that no branch of $\zeta(s)^{\zeta_m^k}$ is holomorphic in a neighbourhood of $s = 1$. □

Let $m > 2$ and $j = 0, 1, \ldots, m-1$. From (4) we get

$$R_{m,j}(x) = \frac{1}{m} \sum_{k=1}^{m-1} \zeta_m^{-jk} S_{m,k}(x) - \frac{\{x\}}{m},$$

where $\{x\}$ denotes the fractional part of x. In light of Lemma 4.2, to obtain that $R_{m,j}(x)$ is not $o(x^\alpha)$ for any $\alpha < 1$, it suffices to show that the generating function of $R_{m,j}(x) + \{x\}/m$, which is

$$\sum_{k=1}^{m-1} \zeta_m^{-jk} L_{m,k}(s),$$

cannot be analytically continued to a holomorphic (single–valued) function in the half plane $\Re s > \alpha$.

Remark 4.4. We can quickly obtain the result for at least two of the error terms as follows. For $k = 1, 2, \ldots, m-1$, using (3) we have

$$S_{m,k}(x) = \sum_{j=0}^{m-1} \zeta_m^{jk} R_{m,j}(x).$$

By Lemma 4.2 and Theorem 4.3, $S_{m,k}(x)$ is not $o(x^\alpha)$ for any $\alpha < 1$, so that at least one of the error terms $R_{m,j}(x)$ is not $o(x^\alpha)$, which is the above mentioned result of Kubota and Yoshida. From (3) with $k = 0$, we obtain

$$\sum_{j=0}^{m-1} R_{m,j}(x) = S_{m,0}(x) - x = -\{x\}.$$

This shows that it is impossible that all but one of the error terms $R_{m,j}(x)$ are $o(x^\alpha)$ for an $\alpha < 1$.

We now proceed with the proof of the main result of this section.

Proof of Theorem 4.1. Let $1/2 < \alpha < 1$ and let $c_1, c_2, \ldots, c_{m-1} \in \mathbb{C}^*$. We shall prove that the linear combination

$$f(s) := \sum_{k=1}^{m-1} c_k L_{m,k}(s)$$

cannot be analytically continued to a holomorphic (single–valued) function in the half plane $\Re s > \alpha$. Suppose to the contrary that it can and assume for now that $L_{m,1}(s), L_{m,2}(s), \ldots, L_{m,m-1}(s)$ are linearly independent over $\mathbb{C}$, which shall

be shown later. Let C denote a closed loop in the half plane $\Re s > \alpha$ winding around $s = 1$ once in the positive direction and not around any zeroes of $\zeta(s)$. As pointed out in [4], the analytic continuation of $L_{m,k}(s)$ along C gives us $\exp\left(-2\pi i \zeta_m^k\right) L_{m,k}(s)$. From the holomorphicity assumption on $f(s)$, it follows that the analytic continuation of $f(s)$ along C is $f(s)$ itself. So

$$\sum_{k=1}^{m-1} c_k L_{m,k}(s) = \sum_{k=1}^{m-1} c_k \exp\left(-2\pi i \zeta_m^k\right) L_{m,k}(s),$$

and from the linear independence over $\mathbb{C}$ of the functions $L_{m,k}(s)$, we obtain that $\exp\left(-2\pi i \zeta_m^k\right) = 1$ for $k = 1, 2, \ldots, m-1$. This means $\zeta_m^k \in \mathbb{Z}$ for $k = 1, 2, \ldots, m-1$, a contradiction if $m > 2$.

We are left with proving that $L_{m,1}(s), L_{m,2}(s), \ldots, L_{m,m-1}(s)$ are linearly independent over $\mathbb{C}$. This can be done along similar lines. Suppose they are not linearly independent over $\mathbb{C}$. Let b be the smallest integer such that there exists a nontrivial linear dependence over $\mathbb{C}$ of b different functions $L_{m,k}(s)$, say $L_{m,k_1}(s), L_{m,k_2}(s), \ldots, L_{m,k_b}(s)$ for $0 < k_1 < k_2 < \ldots < k_b < m$. Since the function $L_{m,k}(s)$ are nonzero, we have $b \geq 2$, furthermore

$$L_{m,k_1}(s) = \sum_{n=2}^{b} d_n L_{m,k_n}(s)$$

for some $d_2, \ldots, d_b \in \mathbb{C}^*$. Analytic continuation along C yields

$$\begin{aligned}\exp\left(-2\pi i \zeta_m^{k_1}\right) L_{m,k_1}(s) &= \sum_{n=2}^{b} d_n \exp\left(-2\pi i \zeta_m^{k_1}\right) L_{m,k_n}(s) \\ &= \sum_{n=2}^{b} d_n \exp\left(-2\pi i \zeta_m^{k_n}\right) L_{m,k_n}(s).\end{aligned}$$

By the minimality of b, we have that the $b-1 \geq 1$ functions $L_{m,k_2}(s), \ldots, L_{m,k_b}(s)$ are linearly independent over $\mathbb{C}$, so $\exp\left(-2\pi i \zeta_m^{k_1}\right) = \exp\left(-2\pi i \zeta_m^{k_n}\right)$ for $n = 2, \ldots, b$. This means $\zeta_m^{k_1} - \zeta_m^{k_n} \in \mathbb{Z}$ for $n = 2, \ldots, b$. One easily obtains that the only possibility for this is when $b = 2$ and $(\zeta_m^{k_1}, \zeta_m^{k_2}) = (1/2 + 1/2\sqrt{-3}, -1/2 + 1/2\sqrt{-3})$ or $(\zeta_m^{k_1}, \zeta_m^{k_2}) = (-1/2 - 1/2\sqrt{-3}, 1/2 - 1/2\sqrt{-3})$. Therefore, to complete the proof of the independence result, it suffices to show that $L_{6,1}(s)/L_{6,2}(s)$ and $L_{6,4}(s)/L_{6,5}(s)$ are not constant. To see this, we use the formula $L_{m,k}(s) = \zeta(s)^{\zeta_m^k} G_{m,k}(s)$, which readily gives

$$\frac{L_{6,1}(s)}{L_{6,2}(s)} = \zeta(s) \frac{G_{6,1}(s)}{G_{6,2}(s)}.$$

The function $\zeta(s)$ has a pole at $s = 1$ and $G_{m,k}(1) \neq 0$, since for $\Re s > 1/2$ we have

$$\prod_{k=1}^{m-1} G_{m,k}(s) = \zeta(ms).$$

We conclude that $L_{6,1}(s)/L_{6,2}(s)$ is not constant. The proof of the result for $L_{6,4}(s)/L_{6,5}(s)$ follows similarly. This completes the proof. $\square$

Remark 4.5. In the spirit of prime numbers races, it seems fitting that further study should be taken to investigate the sign changes of $N_{m,j}(x) - N_{m,j'}(x)$ for

$j \neq j'$. For the case $m = 2$ some such investigations have been undertaken; see [2] and the references therein.

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